Probability problems: individual contest

May 2019

1. Let $p \geq 1$ and $f, g \in L^p[0,1]$ such that $\int_0^1 g(y)dy = 0$. Show that

$$\int_0^1 \int_0^1 |f(x) + g(y)|^p dx dy \geqslant \int_0^1 |f(x)|^p dx.$$

2. Consider an urn with p plus balls and m minus balls in it, where m and p are given nonnegative numbers. You are allowed to pick a random ball from the urn or quit the game. If you decide to pick and get a plus ball you gain a dollar; if you get a minus ball you lose a dollar. You can continue the game but picked balls are not replaced. Denote by $V(m,p)$ the expected value of playing the game. Find a recurrence for $V(m,p)$

3.

3-1 Let X_n be increasing, integrable random variables and converges a.s. to $X \in L^1$, show that, for any sigma algebra $\mathcal{G}$, $\mathbb{E}(X_n|\mathcal{G}) \uparrow \mathbb{E}(X|\mathcal{G})$.

3-2 Let $X_n \geq 0$, Show that

$$\liminf_n \mathrm{E}(X_n|\mathcal{G}) \geq \mathrm{E}(\liminf_n X_n|\mathcal{G}).$$

3-3 Let X_n be random variables in L^1 and $X_n \to X$ a.s. with $|X_n| \leq Z$ in L^1. Show that

$$\mathrm{E}(X|\mathcal{G}) = \lim \mathrm{E}(X_n|\mathcal{G}) \text{ a.s. and in } L^1.$$

3-4 Show that, if f convex and $X, f(X)$ integrable, then

$$f(\mathbb{E}(X|\mathcal{G})) \leq \mathbb{E}(f(X|\mathcal{G})).$$

3-5 Show that $\mathcal{G}_1$ and $\mathcal{G}_2$ are independent i.f.f. for all X $\mathcal{G}_2$-mesurable, $\mathrm{E}(X|\mathcal{G}_1) = \mathrm{E}(X)$.